MTH 301: Group Theory

Assignment III: Abelian Groups, Simple Groups, and Sylow's Theorems

Practice assignment

1. Let G be a finite group.

   (a) Show that $g \in Z(G)$ if, and only if $N_G(g) = G$. .

   (b) If $|G| = p^n$, where p is a prime, then show that $Z(G)$ is non-trivial. [Hint: Use the Class Equation.]

   (c) If $|G| = p^2$, where p is a prime, then G must be abelian.

2. Show that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $\gcd(m, n) = 1$.

3. Up to isomorphism, classify all abelian groups of order p^k, where p is a prime and $k > 1$.

4. Up to isomorphism, classify all abelian groups of orders 105, 270, and 320.

5. If P is a Sylow p-subgroup and Q a p-subgroup of a finite group G, then show that
$$N_G(P) \cap Q = P \cap Q.$$

6. The subgroup $[G, G] = \langle S \rangle$ of a group G generated by elements in the set
$$S = \{ghg^{-1}h^{-1} \mid g, h \in G\}$$
is called the *commutator subgroup or the derived subgroup of* G. It is also denoted by G' or $G^{(1)}$. Establish the following assertions.

   (a) $[G, G] \trianglelefteq G$.

   (b) $G/[G, G]$ is an abelian group called the abelianization of G.

   (c) G is abelain if, and only if $[G, G] = \{1\}$.

7. Show that $[S_n, S_n] = A_n$ for $n \geq 3$.

8. Show that $D_{8n} \ncong D_{4n} \times \mathbb{Z}_2$.

9. If $A, B \lhd G$ such that G/A and G/B are abelian, show that $G/A \cap B$ is abelian.

10. If $K \unlhd G$, then show that $[K, K] \unlhd G$.

11. Show that, up to isomorphism, there is a unique non-abelian group of order 10.

12. Show that groups of orders 20 and 30 are non-simple.

13. Show that if $|G| = 60$ and $n_5 > 1$, then G is simple. [Hint: Assume on the contrary that G has a proper normal subgroup H. Then $n_5 = 6$ and for any Sylow 5-subgroup P, $|N_G(P)| = 10$. Now you will need to use the non-simplicity of groups of order 15, 20 and 30.]

Problems for submission

(Due: 06/10/2023)

- Solve problems 5, 7, 11, and 12 from the practice problems above.